Part II Algebraic Topology – Example Sheet 2

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Questions marked * are optional and should only be attempted after non-starred questions.

1. Recall that for based spaces (X, x_0), (Y, y_0), the wedge product $X \vee Y$ is the quotient of the space $X \sqcup Y$ by the relation generated by $x_0 \sim y_0$. What is the universal cover of $S^1 \vee S^2$? Draw a picture.

2. Let X be a Hausdorff space, and G a group acting on X by homeomorphisms, *freely* (i.e. if $g \in G$ satisfies $g \cdot x = x$ for some $x \in X$, then $g = e$) and *properly discontinuously* (i.e. each $x \in X$ has an open neighbourhood $U \ni x$ such that $\{g \in G \mid g(U) \cap U \neq \emptyset\}$ is finite).

   (i) Show that the quotient map $X \to X/G$ is a covering map.

   (ii) Deduce that if X is simply-connected and locally path-connected then for any point $[x] \in X/G$ we have an isomorphism of groups $\pi_1(X/G, [x]) \cong G$.

   (iii) Hence show that for $n \geq 2$ odd and any $m \geq 2$ there is a space X with fundamental group $\mathbb{Z}/m$ and universal cover S^n. *[Hint: Consider S^n as the unit sphere in $\mathbb{C}^k$.]*

3. (i) Show that the groups $G = \langle a, b \mid a \rangle$ and $H = \langle t \mid \rangle$ are isomorphic.

   (ii) Show that the groups $G = \langle a, b \mid ab^{-3}, ba^{-2} \rangle$ and $H = \langle t \mid t^5 \rangle$ are isomorphic.

   (iii) Show that the groups

   $$G = \langle a, b \mid a^3 b^{-2} \rangle \quad \text{and} \quad H = \langle x, y \mid xyxy^{-1}x^{-1}y^{-1} \rangle$$

   are isomorphic. Show that this group is non-abelian and infinite. *[Hint: Construct surjective homomorphisms to appropriate groups.]*

4. Use Seifert-van Kampen to show that the fundamental group of $S^1 \vee S^1$ is a free group on two generators. Deduce that the inclusion $i : S^1 \vee S^1 = (S^1 \times \{*\}) \cup (\{*\} \times S^1) \hookrightarrow S^1 \times S^1$ does not admit a retraction.

5. A covering space is called *normal* if its corresponds to a normal subgroup. Draw pictures of all the connected degree 2 covering spaces of $S^1 \vee S^1$. Show that they are all normal coverings. Now do the same thing for the connected degree 3 covering spaces of $S^1 \vee S^1$. Which of them are normal coverings?

6. Consider $X = S^1 \vee S^1$ with basepoint x_0 the wedge point, which has $\pi_1(X, x_0) = \langle a, b \rangle$ where a and b are given by the two characteristic loops. Describe covering spaces associated to

   (i) $\langle\langle a \rangle\rangle$, the normal subgroup generated by a,

   (ii) $\langle a \rangle$, the subgroup generated by a,

   (iii) the kernel of the homomorphism $\phi : \langle a, b \rangle \to \mathbb{Z}/4$ given by $\phi(a) = [1]$ and $\phi(b) = [3] = [-1]$.

   What are the fundamental groups of these covering spaces?

7. Show that the free group F_2 contains subgroups isomorphic to the free group F_n for any $n > 1$.

8. Let $Y = \mathbb{RP}^2 \vee \mathbb{RP}^2$ and $*$ be the wedge point.

   (i) Show that

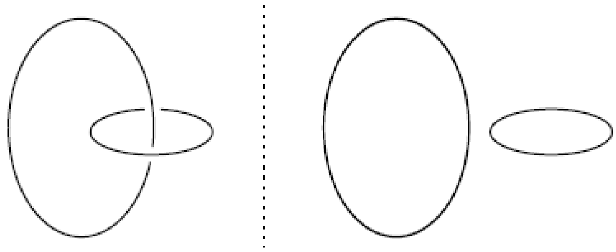
   $$\pi_1(Y, *) \cong \mathbb{Z}/2 * \mathbb{Z}/2 \cong \langle a, b | a^2, b^2 \rangle.$$

   (ii) Describe the covering space of Y corresponding to the kernel of the homomorphism $\phi : \langle a, b \mid a^2, b^2 \rangle \to \mathbb{Z}/2$ given by $\phi(a) = 1$ and $\phi(b) = 0$. Hence show that $\mathrm{Ker}(\phi)$ is isomorphic to $\langle a, b \mid a^2, b^2 \rangle$.

(iii) Draw a picture of the universal cover $\widetilde{Y}$. Deduce that ab has infinite order in $\langle a, b \,|\, a^2, b^2 \rangle$.

9. Show that the Klein bottle has a cell structure with a single 0-cell, two 1-cells, and a single 2-cell. Deduce that its fundamental group has a presentation $\langle a, b \,|\, baba^{-1} \rangle$, and show this is isomorphic to the group G in Q13 of Example Sheet 1.

10. Consider the following configurations of pairs of circles in S^3 (we have drawn them in $\mathbb{R}^3$; add a point at infinity). By computing the fundamental groups of the complements of the circles, show there is no homeomorphism of S^3 taking one configuration to the other.



11* A *graph* G is a space obtained by starting with a set $E(G)$ of copies of the interval I and an equivalence relation $\sim$ on $E(G) \times \{0, 1\}$, and forming the quotient space of $E(G) \times I$ by the minimal equivalence relation containing $\sim$. (That is, it is a space obtained from a set of copies of I by gluing their ends together.) The *vertices* are the equivalence classes represented by the ends of the intervals.

(i) A *tree* is a simply-connected graph. A *star* is a tree with a vertex x_0 such that one end of each edge is attached to x_0. A *leaf* of a tree is a vertex attached to only one edge. Prove that every tree is homotopy equivalent to a star, relative to its leaves.

(ii) If $T \subset G$ is a tree, show that the quotient map $G \to G/T$ is a homotopy equivalence, and that G/T is again a graph. Hence show that every connected graph is homotopy equivalent to a graph with a single vertex. *[You should assume that every graph has a maximal tree.]*

(iii) Show that the fundamental group of a graph with one vertex, based at the vertex, is a free group with one generator for each edge of the graph. Hence show that any free group occurs as the fundamental group of some graph. (We have *not* required that a graph have finitely many edges.)

(iv) Show that a covering space of a graph is again a graph, and deduce that a subgroup of a free group is again a free group.

Comments or corrections to `amk50@cam.ac.uk`.